Раздел 2. Тема 1. Занятие 3.

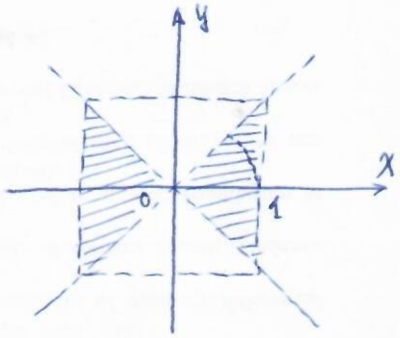
№№ 3951, 3952, 3962, 3966, 3969, 3970, 3971, 3973

№3951. Заменить двойной интеграл однократным $\iint\limits_{x^2+y^2\le 1} f(\sqrt{x^2+y^2})\,dxdy$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad \begin{cases} 0 \le \varphi \le 2\pi \\ 0 \le r \le 1 \end{cases} \Rightarrow$$

$$\iint\limits_{x^2+y^2\le 1} f(\sqrt{x^2+y^2})\,dxdy = \int_0^{2\pi} d\varphi \int_0^1 r f(r)\,dr = 2\pi \int_0^1 r f(r)\,dr.$$

№3952 Заменить двойной интеграл однократным $\iint\limits_{\mathcal{D}} f(\sqrt{x^2+y^2})\,dxdy$, где $\mathcal{D} = \{(x,y):\ |y| \le |x|,\ |x| \le 1\}$



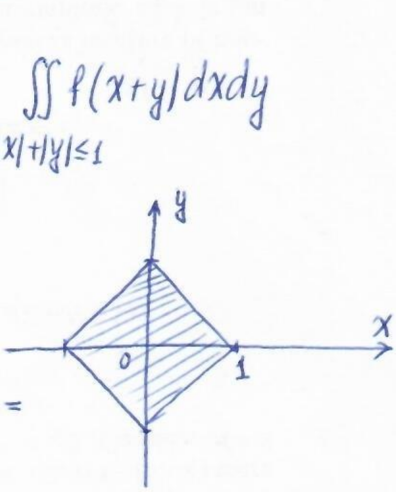
$$\iint\limits_{\mathcal{D}} f(\sqrt{x^2+y^2})\,dxdy = 4 \iint\limits_{\substack{0\le x\le 1\\ 0\le y\le x}} f(\sqrt{x^2+y^2})\,dxdy =$$

$$= \left| \begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix} \right| = 4\int_0^1 r f(r)\,dr \int_0^{\frac{\pi}{4}} d\varphi +$$

$$+ 4\int_1^{\sqrt{2}} r f(r)\,dr \int_{\arccos\frac{1}{r}}^{\frac{\pi}{4}} d\varphi = \pi \int_0^1 r f(r)\,dr + \int_1^{\sqrt{2}} r\left(\pi - 4\arccos\frac{1}{r}\right) f(r)\,dr.$$

№3962 Свести двойной интеграл к однократному $\iint\limits_{|x|+|y|\le 1} f(x+y)\,dxdy$

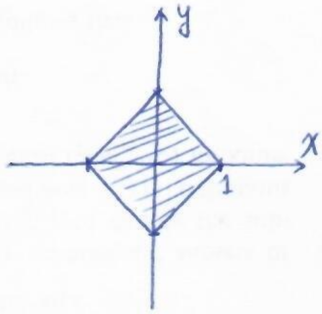
$$\begin{cases} u = y + x \\ v = y - x \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \end{cases} \Rightarrow \ I = \frac{1}{2}$$



$$\begin{cases} -1 \le u \le 1 \\ -1 \le v \le 1 \end{cases} \Rightarrow \iint\limits_{|x|+|y|\le 1} f(x+y)\,dxdy = \frac{1}{2}\int_{-1}^1 du \int_{-1}^1 f(u)\,dv =$$

$$= \int_{-1}^1 f(u)\,du.$$

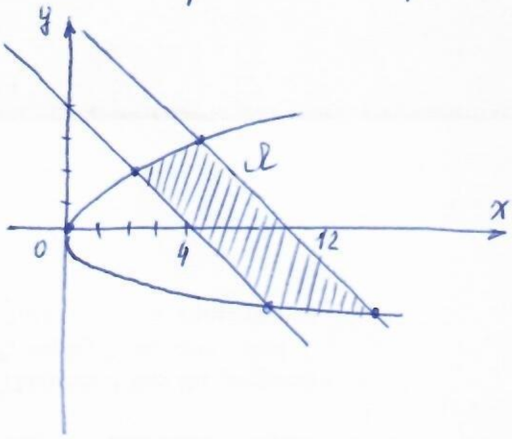
№3966 Вычислить $\iint\limits_{|x|+|y|\le 1} (|x|+|y|)\,dxdy =$



$$= 4\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 1-x}} (x+y)\,dxdy = 4\int_0^1 dx \int_0^{1-x} (x+y)\,dy =$$

$$= 4\int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2} \right) dx = \int_0^1 (2-2x^2)\,dx = 2 - \frac{2}{3} = \frac{4}{3}.$$

№3969 Вычислить $\iint_{\Omega} (x+y)\,dx\,dy$, где область Ω ограничена кривыми $y^2 = 2x$, $x+y=4$, $x+y=12$.



Замена переменных:

$$\begin{cases} u = x+y \\ v = y \end{cases} \Rightarrow I = 1 \Rightarrow \begin{cases} 4 \le u \le 12 \\ -1-\sqrt{2u+1} \le v \le -1+\sqrt{2u+1} \end{cases}$$

Второе соотношение $y^2 = 2x \Rightarrow v^2 = 2(u-v) \Rightarrow$

$$(v+1)^2 = 2u+1 \Rightarrow v = -1 \pm \sqrt{2u+1}.$$

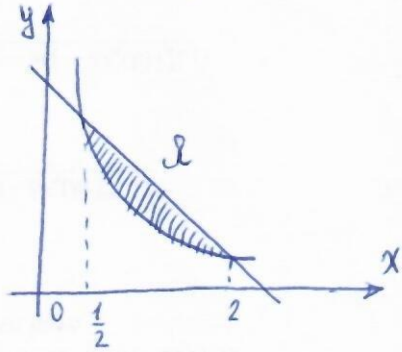
$$\iint_{\Omega} (x+y)\,dx\,dy = \int_4^{12} du \int_{-1-\sqrt{2u+1}}^{-1+\sqrt{2u+1}} u\,dv = \int_4^{12} u \cdot v \Big|_{-1-\sqrt{2u+1}}^{-1+\sqrt{2u+1}} du = 2\int_4^{12} u\sqrt{2u+1}\,du =$$

$$= \int_4^{12} (2u+1)^{\frac{3}{2}} du - \int_4^{12} (2u+1)^{\frac{1}{2}} du = \frac{1}{5}(2u+1)^{\frac{5}{2}}\Big|_4^{12} - \frac{1}{3}(2u+1)^{\frac{3}{2}}\Big|_4^{12} =$$

$$= \frac{1}{5}(5^5 - 3^5) - \frac{1}{3}(5^3 - 3^3) = 5^4 - \frac{3^5}{5} - \frac{5^3}{3} + 3^2 = 634 - \frac{243}{5} - \frac{125}{3} =$$

$$= \frac{9510 - 1354}{15} = \frac{8156}{15} = 543\frac{11}{15}.$$

№3970 Вычислить $\iint_{\Omega} xy\,dx\,dy$, где область Ω ограничена кривыми $xy = 1$ и $x+y = \frac{5}{2}$.



$$\begin{cases} \frac{1}{2} \le x \le 2 \\ \frac{1}{x} \le y \le \frac{5}{2} - x \end{cases} \Rightarrow \iint_{\Omega} xy\,dx\,dy = \int_{\frac{1}{2}}^{2} dx \int_{\frac{1}{x}}^{\frac{5}{2}-x} xy\,dy =$$

$$= \int_{\frac{1}{2}}^{2} x\, \frac{y^2}{2}\Big|_{\frac{1}{x}}^{\frac{5}{2}-x} dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\left(\frac{5}{2}-x\right)^2 - \frac{1}{x^2} \right) x\,dx =$$

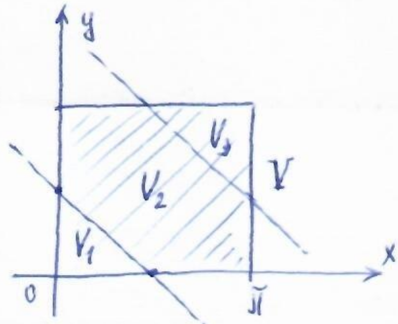
$$= \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(x\left(\frac{5}{2}-x\right)^2 - \frac{1}{x} \right) dx = \frac{1}{2}\int_{\frac{1}{2}}^{2} \left\{ \frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x} \right\} dx =$$

$$= \frac{25}{16}x^2\Big|_{\frac{1}{2}}^{2} - \frac{5}{6}x^3\Big|_{\frac{1}{2}}^{2} + \frac{1}{8}x^4\Big|_{\frac{1}{2}}^{2} - \frac{1}{2}\ln x\Big|_{\frac{1}{2}}^{2} =$$

$$= \frac{25}{16}\left(4-\frac{1}{4}\right) - \frac{5}{6}\left(8-\frac{1}{8}\right) + \frac{1}{8}\left(16-\frac{1}{16}\right) - \frac{1}{2}\left(\ln 2 - \ln\frac{1}{2}\right) = \frac{25}{16}\cdot\frac{15}{4} - \frac{5}{6}\cdot\frac{63}{8} +$$

$$+ \frac{1}{8}\cdot\frac{255}{16} - \ln 2 = \frac{375}{64} - \frac{105}{16} + \frac{255}{128} - \ln 2 = \frac{165}{128} - \ln 2 = 1\frac{37}{128} - \ln 2 .$$

№3971 Вычислить $\iint\limits_{\substack{0\le x\le\pi\\0\le y\le\pi}} |\cos(x+y)|\,dxdy$



для V_1 $|\cos(x+y)| = \cos(x+y)$

для V_2 $|\cos(x+y)| = -\cos(x+y)$

для V_3 $|\cos(x+y)| = \cos(x+y)$

$$\iint\limits_{\substack{0\le x\le\pi\\0\le y\le\pi}} |\cos(x+y)|\,dxdy = \iint\limits_{V_1}\cos(x+y)\,dxdy - \iint\limits_{V_2}\cos(x+y)\,dxdy + \iint\limits_{V_3}\cos(x+y)\,dxdy =$$

$$= 2\iint\limits_{V_1}\cos(x+y)\,dxdy + 2\iint\limits_{V_3}\cos(x+y)\,dxdy - \iint\limits_{V}\cos(x+y)\,dxdy =$$

$$= 4\iint\limits_{V_1}\cos(x+y)\,dxdy - \iint\limits_{V}\cos(x+y)\,dxdy .$$

$$\iint\limits_{V}\cos(x+y)\,dxdy = \int_0^{\pi}dx\int_0^{\pi}(\cos x\cos y - \sin x\sin y)\,dy =$$

$$= \left(\int_0^{\pi}\cos x\,dx\right)^2 - \left(\int_0^{\pi}\sin x\,dx\right)^2 = \left(\overset{0}{\cancel{\sin x}}\Big|_0^{\pi}\right)^2 - \left(\cos x\Big|_0^{\pi}\right)^2 = -4 .$$

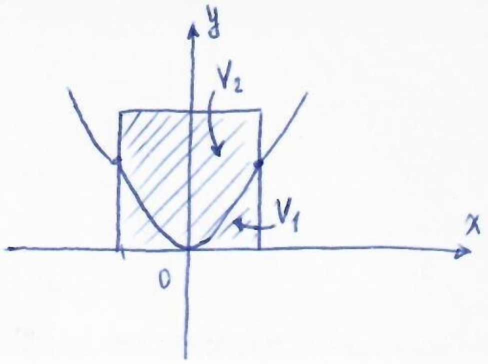
$$\iint\limits_{V_1}\cos(x+y)\,dxdy = \int_0^{\frac{\pi}{2}}dx\int_0^{\frac{\pi}{2}-x}(\cos x\cos y - \sin x\sin y)\,dy =$$

$$= \int_0^{\frac{\pi}{2}}\cos x\,dx\int_0^{\frac{\pi}{2}-x}\cos y\,dy - \int_0^{\frac{\pi}{2}}\sin x\,dx\cdot\int_0^{\frac{\pi}{2}-x}\sin y\,dy = \int_0^{\frac{\pi}{2}}\cos x\sin\left(\frac{\pi}{2}-x\right)dx +$$

$$+ \int_0^{\frac{\pi}{2}}\sin x\left(\cos\left(\frac{\pi}{2}-x\right)-1\right)dx = \int_0^{\frac{\pi}{2}}dx - \int_0^{\frac{\pi}{2}}\sin x\,dx = \frac{\pi}{2} + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 .$$

$$\iint\limits_{\substack{0\le x\le\pi\\0\le y\le\pi}} |\cos(x+y)|\,dxdy = 4\left(\frac{\pi}{2}-1\right) + 4 = 2\pi .$$

№3973 Вычислить: $\displaystyle\iint\limits_{\substack{|x|\le 1\\ 0\le y\le 2}} \sqrt{|y-x^2|}\,dx\,dy =$



$$= 2\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le 2}} \sqrt{|y-x^2|}\,dx = 2\iint\limits_{V_1}\sqrt{x^2-y}\,dx\,dy + 2\iint\limits_{V_2}\sqrt{y-x^2}\,dx\,dy.$$

$$\iint\limits_{V_1}\sqrt{x^2-y}\,dx\,dy = \int_0^1 dx\int_0^{x^2}\sqrt{x^2-y}\,dy = -\frac{2}{3}\int_0^1 (x^2-y)^{\frac{3}{2}}\Big|_0^{x^2}dx = \frac{2}{3}\int_0^1 x^3\,dx = \frac{1}{6}x^4\Big|_0^1 = \frac{1}{6}.$$

$$\iint\limits_{V_2}\sqrt{y-x^2}\,dx\,dy = \int_0^1 dx\int_{x^2}^{2}\sqrt{y-x^2}\,dy = \frac{2}{3}\int_0^1 (y-x^2)^{\frac{3}{2}}\Big|_{x^2}^{2}dx = \frac{2}{3}\int_0^1 (2-x^2)^{\frac{3}{2}}dx$$

$$\int\sqrt{(2-x^2)^3}\,dx = \frac{1}{4}\left(x\sqrt{(2-x^2)^3} + 3x\sqrt{2-x^2} + 6\arcsin\frac{x}{\sqrt{2}}\right) + C \Rightarrow$$

$$\int_0^1\sqrt{(2-x^2)^3}\,dx = \frac{1}{4}\left(1+3+6\arcsin\frac{1}{\sqrt{2}}\right) = 1+\frac{3}{2}\cdot\frac{\pi}{4} = 1+\frac{3\pi}{8} \Rightarrow$$

$$\iint\limits_{\substack{|x|\le 1\\ 0\le y\le 2}}\sqrt{|y-x^2|}\,dx\,dy = 2\cdot\frac{1}{6} + 2\cdot\frac{2}{3}\left(1+\frac{3\pi}{8}\right) = \frac{1}{3}+\frac{4}{3}+\frac{\pi}{2} = \frac{5}{3}+\frac{\pi}{2}.$$